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MULTIVALUED SYNTHESIS FOR ONE CLASS OF CONTROLLABLE SYSTEMS†

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The problem of minimizing the terminal functional on the trajectories of a controllable system which is described by a differential inclusion is considered. A relation is derived which is a function of the optimal result. Multivalued synthesis in the form of a differential inclusion is defined, all the solutions of which are optimal trajectories. A method of approximating the optimal trajectories is indicated.

CONSIDER a controllable system whose behaviour is described by the following differential inclusion:

$$\dot{x} \in F(t, x), \quad x \in R^n, \quad t \in T = [0, \theta] \tag{1}$$

We will assume that

1. $F(t, x)$ is a convex compactum for all $(t, x) \in T \times R^n$.
2. The multivalued transformation $(t, x) \rightarrow F((t, x)$ is continuous with respect to (t, x) and satisfies the local Lipschitz condition with respect to x, i.e.

$$\alpha(F(\tau, y), F(t, x)) \rightarrow 0 \text{ as } (\tau, y) \rightarrow (t, x)$$
$$\alpha(F(t, x_1), F(t, x_2)) \leqslant \lambda(L) \| x_1 - x_2 \|; \quad (t_i, x_i) \in L, \quad i = 1, 2$$

where $L \subset T \times R^n$ is any bounded region, $\|\cdot\|$ is the Euclidean norm, $\lambda(L)$ is a constant and $\alpha(\cdot, \cdot)$ is the Hausdorff spacing.

3. A constant $c > 0$ exists such that

$$\max\{\| f \|: f \in F(t, x)\} \leqslant c(1 + \| x \|)$$

Henceforth we will assume that conditions 1–3 are satisfied.

We mean by the solution of system (1) the absolutely continuous function $x(t)$ which satisfies Eq. (1) almost everywhere.

Suppose that in the solutions of system (1) it is required to minimize the functional

$$\sigma = \sigma(x(\theta)) \tag{2}$$

where σ: $R^n \rightarrow R^1$ is a function which satisfies the local Lipschitz condition. We will denote by $X_1(t_*, x_*)$ the set of all solutions of Eq. (1) with initial conditions $x(t_*) = x_*$. Suppose $f \in R^n$. We will assume that

$$c(t_*, x_*) = \min\{\sigma(x(\theta)): x(\cdot) \in X_1(t_*, x_*)\}$$
$$\partial_* c(t, x) \mid (f) = \liminf_{\delta \rightarrow 0+} [c(t + \delta, x + \delta f) - c(t, x)]\delta^{-1}$$
$$F_0(t, x) = \{f_* \in F(t, x): \min_{f \in F(t,x)} \partial_* c(t, x) \mid (f) = \partial_* c(t, x) \mid (f_*)\}$$
$$S = \{(t, x) \in T \times R^n: F_0(t, x) \text{ is not convex}$$

The function $c(t, x)$ is called the function of the optimal result (FOR) for problem (1), (2). The trajectory $x_0(\cdot) \in X_1(t_*, x_*)$ is said to be optimal for the initial position (t_*, x_*) if $c(t_*, x_*) = \sigma[x_0(\theta)]$. It can be shown that the function $c(t, x)$ satisfies the local Lipschitz condition with respect to (t, x), i.e.

$$\| c(t_1, x_1) - c(t_2, x_2) \| \leqslant \lambda(L)[\, | t_1 - t_2 | + \| x_1 - x_2 \| \,], \quad (t_i, x_i) \in L, \quad i = 1, 2$$

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where $L \subset T \times R^n$ is any bounded region and $\lambda(L)$ is a constant. Hence, the number $\partial_* c(t,x)|(f)$ is always finite.

We will denote by Lip the set of all functions ω: $T \times R^n \to R^1$ which satisfy the local Lipschitz condition. Using well-known results [1–5] it can be shown that the following theorem holds.

Theorem 1. In order that the function $c(\cdot) \in \text{Lip}$ should be a function of the optimal result for problem (1), (2) it is necessary and sufficient that the following conditions be satisfied:

$$\min_{f \in F(t,x)} \partial_* c(t,x) \mid (f) = 0, \quad c(\theta, x) = \sigma(x), \quad \forall (t,x) \in [0,\theta) \times R^n \tag{3}$$

Relation (3) replaces Bellman's equation for problem (1), (2).

We will now define the multivalued synthesis for problem (1), (2) and consider the problems of its existence.

Definition 1. Suppose $(t,x) \to G(t,x) \subset R^n$ is a compact-valued multivalued transformation semi-continuous from above. We will call it a multivalued synthesis for the control problem (1), (2) if the set of solutions of the equation

$$x^{\cdot} \in G(t,x), \quad x(t_*) = x_*, \quad \forall (t_*, x_*) \in T \times R^n \tag{4}$$

is non-empty and all the solutions of Eq. (4) are optimal trajectories for the initial position (t_*, x_*).

We will assume that the following condition is satisfied.

Condition 1. For the FOR see (t,x) the function $(t,x) \to \partial_* c(t,x)|(f)$ is semi-continuous from below at each point $(t,x) \in [0,\theta) \times R^n$ for any $f \in R^n$.

Condition 1 is satisfied, for example, if the function σ is continuously differentiable, while the right-hand side of Eq. (1) has the form $F(t,x) = \text{co}\{f(t,x,u)\colon u \in P\}$, where co is a convex hull, P is compactum, and the function f is continuous over the set of variables and is continuously differentiable with respect to x.

When Condition 1 is satisfied the multivalued transformation $(t,x) \to F_0(t,x)$ is semi-continuous from above. In fact, suppose $(t_k, x_k) \to (t_*, x_*)$, $f_k \to f_*$, $f_k \in F_0(t_k, x_k)$. We will show that $f_* \in F_0(t_*, x_*)$. By the definition of F_0 and by virtue of conditions (3) we have

$$\partial_* c(t_k, x_k) \mid (f_k) = 0 \tag{5}$$

For each (t,x) the function $f \to \partial_* c(t,x)|(f)$ satisfies the Lipschitz condition. Hence, taking Condition 1 into account as well as relations (3) and (5) we have $\partial_* c(t_*, x_*)|(f_*) = 0$. Consequently $f_* \in F_0(t_*, x_*)$.

For the FOR $c(t,x)$ we will denote by $\partial c(t_*, x_*)$ the set of Clarke subgradients of the function $c(t,x)$ at the point (t_*, x_*).

Theorem 1. Suppose Condition 1 is satisfied and also

$$\min_{z \in \partial c(t,x)} \max_{f \in F_0(t,x)} \langle z, (1,f) \rangle \leq 0, \quad \forall (t,x) \in [0,\theta) \times R^n \tag{6}$$

Then the transformation $(t,x) \to \text{co} F_0(t,x)$ is a multivalued synthesis for problem (1), (2).

Proof. When Condition 1 is satisfied the following equation holds:

$$\partial_* c(t,x) \mid (f) = \min\{\langle z, (1,f) \rangle\colon z \in \partial c(t,x)\} \tag{7}$$

Suppose $(t_*, x_*) \in [0,\theta) \times R^n$. Consider the differential inclusion

$$x \in \text{co}\, F_0(t,x), \quad x(t_*) = x_* \tag{8}$$

The right-hand side of the first relation in (8) is convex, compact-valued and semi-continuous from above with respect to (t,x). Hence, using well-known theorems of existence [4], we conclude that Eq. (8) has a solution which satisfies the condition $x(t_*) = x_*$. For any solution $x(t)$ of Eq. (8) the function $\varphi(t) = c(t, x(t))$ satisfies the Lipschitz condition. Hence, by Rademacher's theorem it is differentiable almost everywhere. For almost all $t \in [t_*, \theta]$, taking into account the theorem on the minimax, we have

$$\varphi^{\cdot}(t) = \partial_* c(t, x(t)) \mid (x^{\cdot}(t)) \leq \max_{f \in \text{co} F_0(t,x(t))} \partial_* c(t,x)) \mid (f) = \max_{f \in \text{co} F_0(t,x(t))} \min_{z \in \partial c(t,x(t))} \langle z, (1,f) \rangle =$$

$$= \min_{z \in \partial c(t,x(t))} \max_{f \in F_0(t,x(t))} \langle z, (1,f) \rangle \leq 0$$

Consequently, all the trajectories of Eq. (8) are optimal.

Theorem 1 is applicable, generally speaking, in cases when the surface of non-convexity S of the set $F_0(t, x)$ is a concentrating surface for the system $x^\circ \in \operatorname{co} F_0(t, x)$ (in Isaacs's terminology [6]). When this surface is a scattering surface, inequality (6), generally speaking, will not be satisfied.

For example, for the system

$$x_1^\cdot = x_2,\quad x_2^\cdot = u,\quad |u| \leq 1,\quad t \in [0, 1],\quad \sigma(x_1, x_2) = -(x_1)^2 \tag{9}$$

We will have $c(t, x) = -[K \pm \frac{1}{2}(t-1)^2]^2$, $K = K(t, x) = x_1 + (1-t)x_2$, where the plus sign is taken when $K \geq 0$, and the minus sign is taken when $K < 0$. Therefore,

$$F_0(t, x) = \left\| \begin{matrix} x_2 \\ \zeta \end{matrix} \right\|, \quad \zeta = \begin{cases} 1, & K > 0 \\ -1, & K < 0 \\ -1, 1, & K = 0 \end{cases}$$

We can verify that for this example inequality (6) is violated on the surface $K = 0$.

We will introduce the following condition.

Condition 2. For each $(t_*, x_*) \in T \times R^n$, among the solutions of Eq. (8) a solution $x(t)$ will exist such that the set $\{t \in [t_*, \theta]: (t, x(t)) \in S\}$ has a zero Lebesque measure.

It can be verified that the solution $x(t)$ of Eq. (6), which satisfies Condition 2, is also a solution of the equation $x^\circ \in F_0(t, x)$ and all the solutions of the equation $x^\circ \in F(t, x)$ are optimal for all initial positions (t_*, x_*). The following theorem holds.

Theorem 2. Suppose Conditions 1 and 2 are satisfied. Then the transformation $(t, x) \to F_0(t, x)$ is a multivalued synthesis for problem (1), (2).

Theorem 2 is applicable in cases when S is a scattering surface. Condition 2 is satisfied for example (9). Hence, the transformation $(t, x) \to F_0(t, x)$ constructed is a multivalued synthesis.

It can be seen from Theorems 1 and 2 that for the FOR $c(t, x)$ for finding optimal trajectories, the problem of approximating the solutions of the equation $x^\circ \in \operatorname{co} F_0(t, x)$ is of interest. We will indicate one method of making this approximation. Suppose $(t_*, x_*) \in [0, \theta) \times R^n$, $\Delta = \{t_* = \tau_0 < \tau_1 < \ldots < \tau_N = \theta\}$ is a splitting of the section $[t_*, \theta]$, $\xi = \{\xi_0, \xi_1, \ldots, \xi_{N-1}\}$, where ξ_i is an n-dimensional vector for all i. We will put

$$\operatorname{diam} \Delta = \max_i(\tau_{i+1} - \tau_i), \quad |\xi| = \max_i \|\xi_i\|$$

Consider the step-by-step differential inclusion

$$x^\cdot(t) \in F_0(\tau_i, x(\tau_i) + \xi_i), \quad \tau_i \leq t < \tau_{i+1}, \quad x(t_*) = x_* \tag{10}$$

We will denote by $X_2(t_*, x_*)$ the set of absolutely continuous function $x(\cdot)$: $[t_*, \theta] \to R^n$ such that

$$x(t) = \lim_{k\to\infty} x_k(t) \qquad (t \in [t_*, \theta])$$

where $x_k(t)$ is an absolutely continuous solution of Eq. (10) for the sequences Δ_k, ξ^k, such that

$$\lim_{k\to\infty} \operatorname{diam} \Delta_k = \lim_{k\to\infty} |\xi^k| = 0$$

The following assertion holds: for all $(t_*, x_*) \in [0, \theta) \times R^n$ the bundle $X_2(t_*, x_*)$ is identical with the bundle of all the solutions of Eq. (8). Hence, all the solutions of Eq. (8) are approximated by the solutions of the step-by-step differential inclusion (10).

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